SUMMARY OF RESEARCH

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1. Introduction

My mathematical research is centered around the study of the geometry and topology of manifolds and singular spaces in terms of various categories of locally defined objects. Such are, for example, constructible sheaves, $\mathcal{D}$-modules and, more generally, (sheaves of) modules over more general, usually non-commutative, algebras.

The theory of $\mathcal{D}$-modules is just a coordinate-free way of describing systems of linear partial differential equations. The relationship of holonomic $\mathcal{D}$-modules (overdetermined systems) to topology goes back to Gauss and Riemann and is now known as the Riemann-Hilbert correspondence which relates holonomic $\mathcal{D}$-modules to constructible sheaves (topology). In this way holonomic $\mathcal{D}$-modules enter the study of the topology of singular spaces. This connection has lead to great advances in representation theory, certain aspects of which have become synonymous with the study of $\mathcal{D}$-modules on flag varieties and related spaces, as well as the topology of the singularities of Schubert varieties.

The number of independent solutions of certain natural systems of partial differential equations on a manifold constitutes important topological invariants; examples of these are the Euler characteristic and the Todd genus. Calculations of this type belong to the scope of index theory which, roughly speaking, expresses the number of independent solutions in terms of the intersection theory on the cotangent bundle of the manifold, and includes results ranging from the Gauss-Bonnet formula to the Riemann-Roch and Atiyah-Singer index theorems. All of these, at least in the analytic setting, are subsumed by the Riemann-Roch type theorem for $\mathcal{D}$-modules (or, rather, elliptic pairs of P. Schapira and J.-P. Schneiders) to the proof of which I have contributed. The simplest example of this is the case of a holonomic $\mathcal{D}$-module (due, originally, to M. Kashiwara) in which the number of independent solutions of a holonomic system is calculated as the intersection number of the characteristic cycle of the holonomic $\mathcal{D}$-module (which is a conic Lagrangian subvariety of the cotangent bundle) with the zero-section of the latter – a generalization of the celebrated theorem of Hopf which equates the Euler characteristic of a manifold with the number of zeroes of a vector field on it.

In fact, the above mentioned Riemann-Roch type theorem for elliptic pairs is a consequence of a more general result concerning deformation quantization of symplectic manifolds. The link between $\mathcal{D}$-modules and symplectic geometry is quite direct: $\mathcal{D}$-modules are naturally microlocal (i.e. local on the cotangent bundle) objects, and the cotangent bundle of a manifold is a local model of a symplectic manifold.

The general index formula for symplectic deformation quantization renders itself naturally to the context, language and methods of non-commutative geometry. These extend the traditional applications of differential calculus (symmetries, de Rham theory, characteristic classes, deformation theory) to the context of non-commutative rings and, more generally, categories by making systematic use of Hochschild and cyclic (co)homology. The non-commutative point of view, beside being very natural even in the context of "commutative" geometry, has become the natural environment for a host of issues in theoretical physics (mirror symmetry, for example).

In the following sections I describe more specifically some of the results of my past research as well my current interests and plans for the future.

2. Past research

2.1. **Schubert Calculus.** In [BE2] and [BE3], in collaboration with S. Evens, I studied the intersection theory on the flag variety of a complex reductive Lie group in the context of generalized cohomology theories with complex orientation. The results of [BE2] and [BE3] generalize those of J. Bernstein, I.M. Gel'fand and S.I. Gel'fand ([BGG]) who treated the case of ordinary cohomology and those of B. Kostant and S. Kumar ([KK]) who treated the case of unitary K-theory. The main result is a new algorithm for calculating the cup-product in (complex-oriented) generalized cohomology of the flag variety which depends only on the formal group law associated with the cohomology theory.

2.2. **Characteristic cycles.** D. Kazhdan and G. Lusztig conjectured that the characteristic cycles of simple holonomic $\mathcal{D}$-modules on Schubert varieties in a (classical) flag manifold are irreducible. Jointly with M. Finkelberg and V. Lunts I verified this conjecture for Schubert varieties in a Grassmannian. This result implies the the conjecture for a large class of Schubert varieties in flag manifolds.

2.3. **Fundamental groups of compact Kähler manifolds.** In [ABR], in collaboration with D. Arapura and M. Ramachandran I showed that the fundamental group of a compact Kähler manifold has at most one end thus answering the question posed by F.E.A. Johnson and E.G. Rees in [JR]. This implies, in particular, that the fundamental group of a compact Kähler manifold cannot be a free product.

2.4. **Θ-divisors on Jacobians.** In [BB] with J.-L. Brylinski we studied the intersection cohomology of the Θ-divisor of a non-hyperelliptic curve. We observe that in this case the Abel-Jacobi map is a small resolution of singularities of the Θ-divisor show that the characteristic cycle of the intersection cohomology sheaf of the Θ-divisor is irreducible. In addition we showed that, for curves of even genus, the action of the involution induced by the Serre duality on the intersection cohomology of the Θ-divisor does not preserve the algebra structure induced by the Abel-Jacobi isomorphism with the cohomology of the suitable symmetric power of the curve. Applying the Lefschetz fixed point formula to the action of the involution on intersection cohomology of the Θ-divisor we deduced the classical formula for the number of odd Θ-characteristics.

2.5. **Hodge theory.** In [BSY] with M. Saito and B. Youssin we take the first step toward establishing the link between the L^2 Hodge theory and the theory of polarized Hodge modules of M. Saito. We identify the properties of a filtered differential complex which imply that it is isomorphic to the de Rham complex of a filtered $\mathcal{D}$-module. These properties are encapsulated in the definition of a filtered perverse complex. We show that the de Rham complex of a Hodge module of M. Saito is a differential filtered perverse complex.

2.6. **Deformation quantization and index theory.** In [BNT1] (see [BNT2] and [BNT3] for a detailed exposition) jointly with R. Nest and B. Tsygan I proved the conjecture of P. Schapira and J.-P. Schneiders ([SS], Conjecture 8.5) concerning the microlocal Euler class of a perfect complex of $\mathcal{D}$-modules on a complex manifold. The importance of the microlocal Euler class lies in the fact that it plays the central role in the index formula for elliptic pairs which may be considered as a generalization of the Riemann-Roch and the Atiyah-Singer index theorems.

2.7. **Intersection cohomology for convex polytopes.** In [BL1], in collaboration with V. Lunts, I developed a theory of "equivariant perverse sheaves on the associated toric variety" for general (i.e. not necessarily rational) fans. The theory is "elementary" but sufficiently powerful to embrace such phenomena as the decomposition theorem and Verdier duality among others. In addition, the theory leads in a natural way to certain numerical invariants of convex polytopes which we conjectured to coincide with the generalized h- and g-numbers defined by R. Stanley. We showed that the above conjecture is a consequence of the stronger conjecture which says that the Hard Lefschetz Theorem for intersection cohomology or, rather, its analog in our theory holds. Previously, this result could only be formulated for simple polytopes and was proved by P. McMullen. The general Hard Lefschetz Theorem was recently proved by K. Karu. In [BL2] we give a strengthening of his results. These results, combined, imply among others, the Stanley's conjectures on the unimodality of the generalized h-vector and provide a powerful tool for the study of general convex polytopes which is essentially equivalent to the applications of the methods of algebraic geometry to rational ones via the connection with toric varieties.

2.8. **Courant and Vertex algebroids.** Roughly speaking, Courant and vertex algebroids on manifolds play the same role in field theory analogous to that of the Lie algebroids in differential geometry. The difference between the two notions is the difference between the classical and the quantum fields. Some examples of these objects are well-known in representation theory: central extensions of loop algebras and the Virasoro algebra arise as global sections of vertex algebroids.

In the preprint [B] I gave a classification of certain Courant algebroids (Courant extensions of transitive Lie algebroids). Using this classification I calculated the class of the stack of algebras of chiral differential operators ([GMS]) with the aid of a global coordinate-free construction of the canonical (and unique) vertex algebroid which generates the chiral de Rham complex of [MSV].

2.9. **Deformation quantization of twisted forms.** Traditionally, deformation quantization studies (formal) non-commutative deformations of the structure sheaf of a manifold in the class of sheaves of algebras. From the point of view of non-commutative geometry however it is more natural to consider the (larger) class of algebroid stacks.

Loosely speaking, this is due to Morita invariance of (formal) deformation theory, which is to say that the (formal) deformation theory of an algebra depends not on the algebra, but on the linear category with one object with the algebra comprising the endomorphism ring of this object. A category equivalent to one of this type is called an algebroid.

Algebroid stacks are, speaking somewhat informally, algebroid-valued sheaves. Sheaves of algebras are examples of algebroid stacks. Azumaya algebras can be viewed as algebroid stacks locally equivalent to the structure sheaf of a variety.

In [BGNT1], [BGNT3] we give a classification of algebroid stacks which are locally equivalent to algebras arising from deformation quantization. In particular, as we show in [BGNT2], the (formal) deformaton theory of an Azumaya algebra is canonically equivalent to that of the structure sheaf of a manifold.

In [BGNT5] we generalize our results to algebroid stacks on étale groupoids with applications to deformations of twisted convolution algebras. Future results will also include a variant of the formality theorem which compares classification of quantizations to classification of their quasi-classical limits (which reduce to Poisson structures in the usual case). The quasi-classical limits of algebroid stacks are related to "twisted Poisson structures" which have drawn attention in the recent past and belong naturally to the realm of Courant algebroids.

2.10. **Higher Riemann-Roch theorems.** The Riemann-Roch theorem for families admits various refinements. One such refinement, for example, identifies the determinant (line bundle) of the direct image. A "higher" version of the determinant line bundle would be a determinental (higher) gerbe. These are fairly well known and understood for maps of relative dimension one (i.e. families of curves) in which case one has an explicit geometric construction of the determinental gerbe. It is not clear how to associated a higher determinental gerbe to a map of relative dimension greater than one. It is, however, possible to calculate the de Rham characteristic class of this conjectural object. In [BKTV], to a C^∞ fiber bundle with d-dimensional compact fiber and a complex vector bundle on the total space we associate a class in the $d+2$-dimensional de Rham cohomology of the base manifold. Moreover, we give a Riemann-Roch type formula for this class which expresses the latter as a fiberwise integral of the (component of suitable degree) of the usual Riemann-Roch integrand. In the case of relative dimension d equal to one our construction yields the chiral anomaly as expected.

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